From Ratio to Scaling Law: The Mutual Properties of Music and Mathematics, and the Fenophone as a Measured Instrument

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> Draft v0.1 — first full draft. Paper 5 in the Fenophone research series. Target venue: Journal of Mathematics and Music (alternates: Computer Music Journal, Organised Sound).

ABSTRACT

For twenty-five centuries the relationship between music and mathematics has run between two poles. At one pole music *is* mathematics — generated from a model fixed in advance, whether Pythagorean ratio, Rameau's *corps sonore*, the twelve-tone row, or a Xenakian probability distribution. At the other, mathematics is what one *measures* in the sounding phenomenon — Aristoxenus's ear, Vincenzo Galilei's weights, Helmholtz's resonators, Voss and Clarke's 1/f spectra. This essay reads that history as a single convergent story. It argues, first, that the two poles were never reconciled for long: the generative tradition repeatedly attached a mathematical claim to music that it never verified on the output (the blacksmith legend, Kepler's fudged planetary intervals, Rameau's non-existent subharmonics, Xenakis conceding that a formal structure "cannot be confirmed or invalidated," serial music's post-hoc analysis), while the descriptive tradition measured real music rigorously but built no instruments. Second, that the *mutual property* binding the two traditions has itself narrowed and deepened across the centuries — from a static ratio between two magnitudes, through single-scale periodicity and the harmonic spectrum, through finite-group symmetry, through the probability distribution, to scale-invariant, multifractal burst-and-lull: a power-law "symmetry" holding across all time-scales, and one genuinely shared by turbulence, financial volatility, physiology, and music. We then position the Fenophone — a generative instrument whose intensity core is, by construction, the Markov-switching multifractal of Calvet and Fisher — as the point where the two poles fuse under engineering discipline. Its controls are the model's own parameters (the generative pole taken to its limit); the multifractal and 1/f statistics of its emitted note stream are measured on every build with detrended fluctuation analysis, surrogate-tested, and enforced as continuous-integration invariants (the descriptive pole hardened into a release gate). Design target and verification estimator are the same object — Ptolemy's "reason proposes the ratios, perception confirms them on a calibrated monochord," now automated, quantitative, and applied to a scaling law. The instrument extends the tradition by importing a mature, estimable, invertible generative model from a sister domain that *shares* the mutual property, rather than borrowing a metaphor or a noise source — the disciplined modern successor to *musica mundana*'s "same proportion at every scale,"

recast from a metaphysical assertion into a testable invariant. We close by locating what the Fenophone does not settle: whether the structure is *heard* remains the open Aristoxenian question, and the ear keeps the final vote.

1. Introduction

Music and mathematics have been described as siblings so often that the description has gone soft. The claim of this essay is narrower and, I hope, sharper: across their long shared history the two disciplines have been bound by a small number of genuine *mutual properties* — abstract structures that are simultaneously mathematical objects and features of music — and the way those properties have been *used* divides cleanly into modes that have almost never been combined well. Naming the modes, tracing the properties, and locating one instrument precisely within them is the work of the paper.

Three modes recur. The first is **speculative**: mathematics as the hidden order of the cosmos, of which audible music is a shadow — the music of the spheres, from Pythagoras to Kepler. The second is **generative** (or prescriptive): a mathematical object fixed in advance *produces* the music, or licenses what may be written — Pythagorean tuning, Rameau's derivation of harmony from the overtone series, Schoenberg's row, Xenakis's stochastic distributions. The third is **descriptive** (or analytic): mathematics is what one *measures* in the sounding phenomenon, with the ear or the data as the final arbiter — Aristoxenus, the acoustics of Helmholtz, the 1/f spectra of Voss and Clarke. The generative and descriptive modes correspond to two enduring poles, and the deepest recurring fault line in the whole history — set already by the ancient dispute between the rationalist *kanonikoi* and the empiricist *harmonikoi* — is whether number *prescribes* music or is *measured from* it.

Two observations organize the rest of the paper. The first is a persistent failure. Whenever the generative mode has advanced a mathematical claim about music, it has tended to advance it *unchecked against the sounding output*. The Pythagoreans attributed the consonances to a blacksmith's hammer-weights that produce no such thing (§2). Kepler's planetary "harmonies" fit the data only within a fudge tolerance, and one planet-pair fails even that (§3). Rameau derived the major triad from the harmonic series and then needed a physically non-existent _sub_harmonic series to reach the minor triad (§3). Xenakis composed a first-order Markov texture and conceded that its hypothesized second-order sonority "cannot be confirmed or invalidated" (§5). Serial and set-theoretic structure is asserted by construction and can be checked only by a human analyst reading the score afterward (§5). The mutual property was, again and again, *assumed* rather than *measured on what was actually emitted*.

The second observation is a convergence. The mutual property at the centre of the relationship has not been static; it has narrowed and deepened. It begins as **ratio** — a relation between two magnitudes, two string-lengths (§2). It becomes **periodicity** and then the **harmonic spectrum** — a relation among the components of one signal at one scale (§4). In the twentieth century it becomes, on one branch, **finite-group symmetry** — exact, discrete invariance under transposition, inversion, retrograde (§5) — and on another, the **probability distribution** — the statistical law from which events are drawn (§5). And in the last branch it becomes **scale invariance**: a power-law self-similarity holding not between two magnitudes or within one scale but *across all scales at once*, and made precise as multifractality (§6). This is the funnel down which the paper travels — from ratio to scaling law — and its narrow end is where a particular instrument lives.

That instrument is the Fenophone, a browser-based generative instrument whose intensity core is, by construction, the Markov-switching multifractal (MSM) of Calvet and Fisher, and whose statistical properties are enforced in continuous integration. Its full technical case is made in four companion papers: the instrument and its measured spectral landscape (Atlas 2026a); the theorems and statistical contracts that gate it (Atlas 2026b); the MSM-as-model-of-musical-intensity argument and a perceptual research programme (Atlas 2026c); and the analysis of how scaling exponents survive musical quantization (Atlas 2026d). This paper does something the other four deliberately do not: it locates the instrument in the twenty-five-century history sketched above, and argues that the Fenophone is the point at which the generative and descriptive poles finally fuse — where a mathematical model both *generates* the music and has the mathematical claim about its output *measured and enforced* — and that this fusion is the honest modern completion of a synthesis first attempted, and first abandoned, in antiquity.

A note on method and honesty, since the paper's thesis is partly about honesty. The history of music and mathematics is unusually thick with romance — apocryphal discoveries, misattributed inventions, and analogies mistaken for theorems. I have tried to flag the best-known of these where they touch the argument, both because accuracy matters and because the pattern of the myths *is* the argument: they are precisely the cases in which a beautiful mathematical story was told about music and never checked. General surveys of the terrain are given by Fauvel, Flood, and Wilson (2003) and Benson (2007); I lean on them for the parts the narrative only touches.

2. Number in time: antiquity and the two poles

Music enters the Western intellectual record already classified as a branch of mathematics. In the curriculum that Boethius (c. 500 CE) transmitted to the Latin Middle Ages, music is one of the four mathematical disciplines of the *quadrivium*, beside arithmetic, geometry, and astronomy: arithmetic is number in itself, music number in relation and in time, astronomy number in motion (Bower 1989). This is not a metaphor about music being "orderly." It is a claim of literal disciplinary identity, and it held for a thousand years.

The mutual property that grounds the claim is **ratio**. The tradition holds that the consonances correspond to ratios of small whole numbers — the octave 2:1, the fifth 3:2, the fourth 4:3 — drawn from the first four integers, whose sum is the sacred *tetractys*. The doctrine is genuinely ancient and genuinely central; it is also, in its popular form, a museum of misattribution. Almost nothing in it can be safely credited to Pythagoras himself, who left no writings and was, on the best modern reconstruction, a religious and political figure rather than a mathematician (Burkert 1972); the number-philosophy belongs to Philolaus in the fifth century and the rigorous mathematics to Archytas in the fourth. The single most repeated story — that Pythagoras discovered the ratios by weighing a blacksmith's hammers of proportional weight — is physically false, and known to be false: the pitch of a struck body is fixed by the body's own resonance, not by the weight of the hammer, and frequency does not scale with striking weight as the tale requires. Its earliest surviving teller is Nicomachus, around 100 CE, half a millennium after the fact (Barker 1989). The numbers 12:9:8:6 do encode the octave, fifth, and fourth — but as *string lengths*, never as hammer weights.

Two features of the ancient settlement matter for everything that follows. The first is that the ratios were ratios of *length*, not of frequency. The Greeks had no way to measure the rate of a vibration; the identification of pitch with frequency is a seventeenth-century achievement (§3). That the ancient ratios survived the transition is a piece of luck — length is inversely proportional to frequency — but it means the founding doctrine was, strictly, a rule about a measurable geometric quantity that *happened* to track the physics, not a physical explanation of consonance. The doctrine was normative: it said which intervals *should* be admitted, and it enforced that norm through an instrument, the monochord (*kanon*), whose single stretched string, divided at marked ratios, is at once a sound source and a measuring device. The monochord is the first appearance of a theme this paper will end on: an instrument that is also an apparatus, a music-maker that certifies a number.

The second feature is that the settlement was contested from within, and the contest never closed. Against the rationalist *kanonikoi*, who held that reason and ratio judge music and that the ear is a fallible servant, stood Aristoxenus (fl. c. 335 BCE), who founded music as an autonomous science of the phenomena *as heard*. He treated pitch as a continuous magnitude — a geometric line, infinitely divisible — and intervals as quantities judged by hearing and memory, explicitly refusing to reduce them to numerical ratios (Barker 2007). His willingness to speak of dividing a tone into two equal halves is, in the strict ratio framework, an impossibility: Archytas had proved that a superparticular ratio (n+1):n has no rational geometric mean, so the whole tone 9:8 cannot be split into two equal parts within the arithmetic (Barker 1989). The tone could be halved by the ear or bisected on the ruled line; it could not be halved by the integers. Here, at the very origin, is the paper's central dialectic in its purest form: reason prescribes a discrete, exact structure that the continuous, measuring ear declines to obey.

It fell to Ptolemy (c. 150 CE) to attempt the first mature synthesis, and his formulation is worth stating precisely because the Fenophone will reproduce it. Ptolemy rejected *both* naive Pythagorean number-worship, which ignores perception, *and* pure Aristoxenian empiricism, which he thought abandoned reason: rational hypotheses, expressed as ratios, must be set up by reason and then *tested and confirmed by perception* on an accurately calibrated instrument (Barker 1989). Reason proposes; the ear disposes; the monochord is where the proposal is checked. This is the ancient ancestor of "measured, never assumed" — a design target fixed in advance by theory, and a measurement that can refute it. Antiquity treated the two poles as enemies and Ptolemy as a truce; the argument of this paper is that the Fenophone is the truce made rigorous.

One further ancient idea will echo at the funnel's end. The tradition held that the same proportions structure the audible music of instruments (*musica instrumentalis*), the concord of body and soul (*musica humana*), and the order of the cosmos (*musica mundana*) — the inaudible "music of the spheres" that Plato dressed in the world-soul construction of the *Timaeus* (the "lambda" numbers 1, 2, 3, 4, 9, 8, 27, filled by harmonic and arithmetic means into a Pythagorean diatonic scale; no golden ratio and no mysticism beyond the plain Pythagorean procedure of the text). The spheres never literally sounded — Aristotle had already ridiculed the audible reading — and the doctrine is speculative metaphysics, not science. But its *form* is the assertion that one and the same mathematical law holds across radically different scales of being. Stripped of metaphysics and made empirical, that is exactly the claim of scale invariance: the same statistical law at every time-scale. The ancients asserted cross-scale isomorphism as cosmology; the Fenophone measures a cross-scale statistical law as an engineering invariant (§6–§7). The distance between those two is the distance this paper traverses.

3. The empirical turn: from sacred number to measured mechanism

Between roughly 1550 and 1750 the generative doctrine of ancient harmonics collided with measurement, and lost — or rather was forced to re-found itself on physics. The period is both the birth of quantitative acoustics and a catalogue of the ways a mathematical analogy overreaches when it is not checked. It is, for the argument here, the founding case study in "measured, never assumed."

The collision has an arithmetic edge that no amount of theory could file down: the circle of fifths does not close. Twelve pure fifths exceed seven octaves by the Pythagorean comma, 531441:524288, about 23.46 cents; the Pythagorean major third (81:64) exceeds the pure 5:4 third by the distinct syntonic comma, 81:80, about 21.5 cents (the two commas are routinely, and wrongly, conflated). These are not sloppiness in the ear or the instrument; they are number-theoretic facts, $3^{12} \neq 2^{19}$, and they *force* a compromise. Every temperament is an engineered distribution of that unavoidable discrepancy — meantone sacrificing pure fifths for purer thirds (Zarlino's exact 2/7-comma recipe of 1558; quarter-comma the later standard), equal temperament dissolving it into twelve identical steps of the irrational ratio $2^{1/12}$, computed to high precision by Zhu Zaiyu in 1584 and, independently and unpublished until 1884, by Simon Stevin around 1585 (Duffin 2006; on the comma and temperament generally, Benson 2007). The comma is a small, exact model of the entire Fenophone loop: a simple generative rule (stack pure fifths) has an emergent, *measurable* global consequence (it does not close) that must be quantified and reconciled. The design target and its own refutation are written in the same arithmetic.

Against the sacred integers came the experiment. Vincenzo Galilei — lutenist, theorist, and father of Galileo — dismantled the numerology of his own teacher Zarlino by the simple expedient of measuring. Hanging weights from strings, he showed that the "magic" octave ratio is 2:1 only in string *length*: in *tension* it is 4:1, and for pipe volumes 8:1; the fifth's tension ratio is 9:4 (*Dialogo*, 1581). The privileged small integers of the tetractys are therefore an artifact of *which physical quantity one chooses to measure* — not a law of nature (Fauvel, Flood, and Wilson 2003). This is the descriptive pole asserting itself against the generative one with a decisiveness the ancients never mustered: number is contingent on measurement. (Vincenzo did not, contrary to a common claim, invent equal temperament; his 18:17 fret rule is a practical near-equal approximation, not the irrational $2^{1/12}$.)

Once pitch was re-identified as *frequency* — the rate of a vibration, a conception stated clearly by Benedetti around 1563 and expounded by Galileo in 1638, both grounding consonance in the periodic *coincidence* of vibrations rather than in static number — the whole field became measurable, and it was measured. Mersenne (1636) stated the laws of the string (frequency proportional to $\frac{1}{L}\sqrt{T/\mu}$), achieved the first absolute determination of an audible frequency by building strings long enough that their slow vibrations could be counted by eye and then scaling up, and — relevant to a later branch of this history — tabulated the factorial permutations of the scale, 720 melodies from a hexachord, 40320 from eight notes, tables running to 22! and beyond. Sauveur (1701) named the discipline “acoustique,” systematized the harmonic partials and their nodes, and used acoustic beats to fix absolute pitch. Measurement had superseded assertion.

And then, having built the machinery of measurement, the age immediately over-reached with it — which is the part of the story most worth remembering. Rameau (1722) set out to derive the entire grammar of tonal harmony — the triad as primitive, the identity of a chord under inversion, the fundamental bass governing progression — from a single natural fact. (The 1722 *Traité* in fact grounds this in the mathematical division of the string and Zarlino’s *senario*; Rameau re-founded it on the physical overtone series, the *corps sonore*, only after learning of Sauveur’s harmonics, in the works of 1726 and 1737; the popular one-line summary telescopes two distinct phases — Christensen 1987.) The major triad 4:5:6 sits low and clear in the harmonic series of any resonant body. The minor triad does not — and Rameau, to reach it, invoked a descending _sub_harmonic series that no physical body radiates, an inconsistency he himself conceded. Euler (1739), in the same spirit, tried to make agreeableness *computable*: his *gradus suavitatis* assigns to an interval a number-theoretic complexity read off the prime factorization of its ratio ($G(n) = 1 + \sum a_i (p_i - 1)$, so the octave scores 2, the fifth 4, the major triad 9), an elegant early consonance statistic that never fit perception well and rewarded only exact just ratios; the work was ignored as “too mathematical for musicians, too musical for mathematicians” (Lynch 2018). Kepler (1619), in the grandest overreach of all, fitted musical intervals to the ratio of each planet’s maximum and minimum angular velocity as seen from the Sun — but the fit holds only within a tolerance of a diesis (25:24), the Jupiter–Mars pair fails even that, and the genuinely durable result printed in the same book, the third law of planetary motion, is an independent empirical fit that owes nothing to the musical harmonies (a conflation invited by the shared word “harmonic”; the harmonies themselves Kepler grounded not in Pythagorean arithmetic but in which regular polygons are constructible by straightedge and compass).

The lesson of the period is thus double, and the Fenophone inherits both halves. On one hand, the overtone series is the paradigm of a *measurable* *natural invariant imported as a generator*: Rameau's ambition was to make a whole compositional grammar follow from one estimable physical fact, exactly the move (one domain over) that the Fenophone makes with the Calvet–Fisher MSM (§7). On the other, every attractive mapping in this era is caught fudging when checked — Kepler's tolerance, Rameau's phantom undertones, Euler's poor fit, and, a generation later, Newton forcing the spectrum into seven colours to match the diatonic scale. A beautiful mathematical story about music that is never held to the sounding output is the characteristic failure mode of the generative tradition, and it is precisely the failure that a continuous-integration statistical gate is built to prevent (Atlas 2026b).

The same period furnishes one more ancestor, on a different axis. In the English bell-towers, change ringers were building something the mathematicians would not formalize for another 150 years. An "extent" rings all n! permutations of n bells, each exactly once, under a hard constraint — from one row to the next, each bell moves at most one position — and never repeats a row; on seven bells that is 5040 changes. In modern terms this is a Hamiltonian cycle on the Cayley graph of the symmetric group S_n generated by adjacent transpositions, and Fabian Stedman codified constructive methods to achieve it (*Tintinnalogia*, 1668; *Campanalogia*, 1677). White's careful title — "Fabian Stedman: The First Group *Theorist*?", with the question mark — marks the anachronism: the ringers constructed and exhaustively enumerated; they proved no group axioms and had no concept of a group (White 1996). What they had was a *deterministic, integer-exact, exhaustively generated combinatorial structure with a hard checkable invariant* — an extent either is or is not complete — and that is the pre-modern sibling of the Fenophone's deterministic event layer and its pass/fail contracts (§7). Change ringing is generative mathematics whose correctness is a decidable property of the output; it simply predates the machinery that could state the property abstractly.

4. The spectrum: analysis comes of age

If the seventeenth century made consonance measurable, the eighteenth and nineteenth made the sound itself analyzable, and in doing so matured the descriptive pole into something recognizably modern: perceptual qualities reduced to *estimable statistics of a signal*, derived, and tested — sometimes to destruction.

The engine of the transformation was, once again, a musical object: the vibrating string. Taylor (1713) derived its fundamental mode dynamically from mechanics; d'Alembert (1747) wrote the first partial differential equation of mathematical physics for it, the wave equation, whose general solution superposes two travelling waveforms. What followed was a sixty-year controversy — the "arbitrary function" dispute among d'Alembert, Euler, and Daniel Bernoulli over whether a plucked, corner-having string shape could be represented at all, and whether an infinite sum of sine modes was general enough to do it — that enlarged the very notion of a function and drove the birth of rigorous analysis (Wheeler and Crummett 1987). Bernoulli (1753) asserted, on physical and auditory grounds — one hears a fundamental with its overtones — that any motion of the string is a superposition of its sinusoidal modes; but he gave no way to find the amplitudes, and Euler and Lagrange rejected the generality of the claim. The missing machinery arrived from an entirely different problem. Fourier (1822), working on the diffusion of *heat*, supplied explicit integral formulas for the coefficients of a trigonometric series and the bold claim that an essentially arbitrary periodic function equals its sine-and-cosine expansion; Dirichlet (1829) gave the first rigorous convergence conditions, closing the string controversy after eighty years. (Fourier proved no convergence theorem himself, and his tool grew from heat, not from the string — two details the tidy textbook story tends to erase.) The upshot is a correspondence that the rest of acoustics would live on: a signal has two dual representations, a waveform in time and an amplitude spectrum in frequency, and the map between them is explicit and invertible. Time–frequency duality is the mutual property this thread contributes, and the harmonic spectrum is the natural coordinate system it installs.

Ohm (1843) applied the correspondence to the ear: a complex tone is heard as its set of sinusoidal partials, and the ear behaves as a kind of Fourier analyzer, assigning pitch to the fundamental component and (the strong form of the claim) deaf to the relative phases of the partials. Seebeck's siren experiments almost immediately exposed the over-reach — the ear perceives a pitch at the fundamental *period* even when the corresponding spectral component is absent, the "missing fundamental" — and Ohm's law survives only as an approximation (Turner 1977). The dispute seeded physiological acoustics and its master document, Helmholtz's *On the Sensations of Tone* (1863; translated by Ellis, with additions, in 1875 and 1885). Helmholtz operationalized three qualities that antiquity had left metaphysical: **timbre** is the harmonic amplitude spectrum, the vector of partial strengths, which he isolated physically with tuned resonators; **dissonance** is roughness, the beating between partials and difference-tones whose frequencies fall within a critical band (later tied quantitatively to critical bandwidth by Plomp and Levelt 1965); and **hearing** is a physiological resonance mechanism in the cochlea. Rayleigh's *Theory of Sound* (1877) then made "the modes and spectrum of a resonator" a fully calculable object for strings, membranes, plates, and pipes alike, and Ellis (1885) put pitch itself on a logarithmic axis with the cent, $1200 \log_2(f_2/f_1)$, which turns multiplicative frequency ratios into additive measurements, fully rationalizes equal temperament as the geometric division of the octave, and — measured across cultures — showed that scales are conventional rather than dictated by nature.

Two aspects of this maturity bear directly on the Fenophone. The first is the posture, which is exactly the Fenophone's: a single signal statistic serving *simultaneously* as theory, as measurement, and as a falsifiable claim, built into an instrument and tested — and honestly falsified when wrong. Ohm's law was too strong; Helmholtz's specific cochlear mechanism was mechanically wrong, superseded by Békésy's travelling wave and, later, active amplification; the beating account of dissonance is real but partial. The descriptive tradition earns its authority precisely by being the tradition that lets measurement refute the model. Where Helmholtz measured the spectrum of *existing* sound to *explain* perception, the Fenophone makes its target statistic both the design objective of a generative core and a regression gate on the generated output — the estimator promoted from analysis tool to invariant (Atlas 2026a; 2026b). The second is a structural bridge. This thread's mutual property is single-scale: periodicity, the harmonic spectrum, one privileged decomposition into components at fixed frequencies. The Fenophone generalizes it to *time-scale* duality. Where Fourier decomposes a signal into components at fixed scales, the multifractal (singularity) spectrum decomposes it across a continuum of scales, replacing a flat, scale-fixed representation with a scale-free, self-similar one; and where Ellis's cent reads pitch off a log axis, detrended fluctuation analysis reads burst-and-lull structure off a log–log scaling plot (Atlas 2026c). The spectrum was the right idea at the wrong number of scales; the funnel's next turn fixes the number of scales to *all of them*.

5. Structure: symmetry and the process in the twentieth century

The twentieth century made mathematics *constitutive* of music in two distinct programs, and the Fenophone descends from both while repairing the specific weakness each carried.

The first program is **symmetry and group-action**, and it treats the twelve equal-tempered pitch classes as the cyclic group $\mathbb{Z}_{12}$ acted on by finite groups. Schoenberg's method of composing with twelve tones (formulated in 1921) fixes an ordered row and deploys it under four operation-types — prime, inversion, retrograde, retrograde-inversion — each at twelve transposition levels, yielding up to forty-eight forms; those operations form a group of order forty-eight, and transposition with inversion alone generates the dihedral group of order twenty-four acting on the pitch classes. The crucial historical fact is that Schoenberg used *none* of this vocabulary and resisted mathematical readings of his method; the group-theoretic formalization is the work of later theorists, and Josef Hauer had arrived at a twelve-tone practice, with a genuine combinatorial classification of the forty-four hexachord “tropes,” independently and slightly earlier. Babbitt — a mathematics graduate — recast the system as explicit structure, formalizing pitch-class, aggregate, and partition, proving that there are exactly six all-combinatorial source hexachords, and building the time-point system that maps the mod-12 arithmetic of pitch onto rhythm (Babbitt 1962). Messiaen, arriving intuitively rather than algebraically, exploited four exact devices: modes of limited transposition (pitch-class sets invariant under some non-trivial transposition — subsets of $\mathbb{Z}_{12}$ with non-trivial stabilizer), non-retrogradable rhythms (temporal palindromes), and prime-number durations whose coprime cycle lengths realign only at their least common multiple (a 17-value rhythm against a 29-chord cycle in *Liturgie de cristal*, meeting only after 493 units) — the “charm of impossibilities,” a theological rather than a mathematical framing of the same coprimality (Messiaen 1944). Forte (1973) built pitch-class set theory — set classes as orbits of $\mathbb{Z}_{12}$-subsets under the twenty-four-element group of transposition and inversion, the interval-class vector, the Z-relation between distinct classes sharing an interval content — and its combinatorics is exactly orbit-enumeration (Hook 2007). Lewin (1987) generalized the whole apparatus into transformational theory, in which a generalized interval system is a simply transitive group action — a torsor — on a musical space, shifting attention from interval-as-distance to transformation-as-gesture; neo-Riemannian theory then revived Euler’s Tonnetz as a torus acted on by the PLR group, dihedral of order twenty-four, dual (in the precise sense of mutual centralizers in the symmetric group on the twenty-four triads) to the transposition/inversion group (Crans, Fiore, and Satyendra 2009); and Mazzola (2002) pushed the program into category and topos theory. The mutual property is exact, finite, discrete symmetry, and the escalation over the century — ordered permutation, combinatorial set, group action, dual groups on a torus, topos — is one of the most sustained infusions of pure mathematics into any art.

The second program is **the probability distribution**, and it is the Fenophone’s direct ancestor. Its pivot, due chiefly to Xenakis, is to stop writing individual notes and instead specify the *statistical law* that generates them, letting the law of large numbers fix the macroscopic form. Xenakis modeled the glissando velocities of a string cloud on the Maxwell–Boltzmann distribution of a gas (*Pithoprakta*, 1955–56, computed by hand); set event densities by a Poisson law (*Achorripsis*); coded the whole apparatus into the ST program on an IBM 7090 (1961–62); drove textural change with a first-order Markov chain over parameter "screens" (*Analogique* A and B, 1958–59); framed a two-orchestra contest as a zero-sum game with a minimax solution (*Duel*, *Stratégie*); generated macro-form from the rotation group of the cube, order twenty-four, isomorphic to S_4 (*Nomos Alpha*, 1965–66); built scales and rhythms from Boolean combinations of residue classes (sieve theory); and, at the end, drove the waveform itself with bounded second-order random walks (dynamic stochastic synthesis, GENDYN). "Stochastic," he insisted — from the Greek *stokhos*, aim — meant asymptotic evolution toward a goal, the determinism of the *law* over the indeterminacy of the event, and he opposed it sharply to Cageian chance (Xenakis 1971/1992). Alongside Xenakis run Hiller and Isaacson's *Illiac Suite* (1955–57), the first substantial score algorithmically composed by a computer — four movements that are four experiments in rule-based generate-and-test and Markov-chain generation, framed explicitly as science (Hiller and Isaacson 1959; the machine did not merely *play* a tune, as CSIRAC had around 1950, but *composed* — Doornbusch 2004) — and Koenig's Project 1 and Project 2, which generalize serial permutation into parametric selection principles spanning determinism to randomness. This branch also had a running conversation with information theory, which recast musical style as a balance of entropy and redundancy and expectation as the management of uncertainty (Meyer 1957; Youngblood 1958; Moles 1958/1966; Cohen 1962) — the "optimal-complexity," neither-boring-nor-random hypothesis that reappears, quantified, in the Fenophone's own preference-surface experiment (Atlas 2026c).

The Fenophone inherits the ambitions of both programs and closes the specific gap each left open. From the symmetry program it takes *mathematics as constitutive structure* — its player's controls are the parameters of an exact, named mathematical model, exactly as a serial composer's controls are a row and its operations — and its integer, fixed-point, bit-identical event layer echoes the discrete determinism and reproducibility of row and set operations (Atlas 2026b). But it repairs that program's defining weakness, the *post-hoc formalization gap*: serial and set-theoretic structure is asserted by construction and can be verified only by a human analyst reading the score after the fact, the long gap from Schoenberg's practice to Forte's formalization. The Fenophone makes its target statistical invariant a machine-checked release gate on *every build* — the design target and the verification estimator are one object (§7). From the stochastic program it takes the founding move — compose the process, not the notes; its intensity core *is* a Markov-switching stochastic process, and its controls are that process's own parameters, in the exact spirit of Xenakis's densities and Koenig's selection principles (Atlas 2026c). But it repairs *that* program's weakness too, which is the one this paper has tracked throughout: the tradition *trusted* its processes and did not measure them. Xenakis conceded that *Analogique*'s hypothesized second-order sonority "cannot be confirmed or invalidated"; the exotic distribution choices of GENDYN turned out to make little audible difference; the output statistics were, in every case, assumed rather than verified. The Fenophone's output statistics are measured on every build and enforced (§7). It is the completion the stochastic tradition never had.

The stone retrograde: a symmetry coda, measured

Before the funnel narrows to its last turn, one measurement is worth a coda, because it shows that a mutual property of this section's kind — exact, finite, discrete symmetry — is far older than the twentieth century that formalized it, and because it draws, on a single physical object, the boundary between this section's mutual property and the next one's. The object is the unicursal labyrinth: a single non-branching path folded to fill an annulus, every circuit walked exactly once, in the design that runs from the Cretan seven-circuit pattern to the eleven-circuit pavement laid at Chartres around 1200. A labyrinth of this kind is fully specified by its **level sequence** — the order in which the path visits the circuits, an integer sequence such as Chartres's 0 5 2 3 4 1 6 11 8 9 10 7 12 — and on that sequence two operations are defined that every musician in this section would recognize: the **complement** (level $j \mapsto n - j$, the inside-out relabelling) and the **reversal** (walking the path from the centre back out). The classical labyrinths satisfy an exact identity: *the complement of the level sequence equals its reversal.* That property — self-duality — is not an analogy to musical retrograde; it is the **same** $\mathbb{Z}_2$ **group element**, the order-two operation under which a structure read backward is a rigid relabelling of itself read forward, the element Machaut composed into *Ma fin est mon commencement* ("my end is my beginning" — a rondeau whose music is its own retrograde) and that the *L'homme armé* tradition set as the instruction *cancrizat*. The identification is exact, not metaphorical, and it is verified as pure arithmetic in a sealed record of the Feno program, the cross-domain research constellation to which the Fenophone's companion papers belong (Atlas 2026e): reading the labyrinth from the centre out is the numeric complement of reading it from the outside in. A canon in stone.

What raises this from an elegant observation to a measurement is that the labyrinth's design space is *finite and enumerable*. Phillips's combinatorial analysis gives three conditions a level sequence must satisfy to be a valid ("simple, alternating, transit") maze, and under them the designs can be counted outright: exactly 1,828 valid twelve-level designs exist (Phillips 2015; the enumeration is reproduced byte-exactly in the sealed record). Of the 1,828, exactly **72 are self-dual** — Chartres among them. The Cretan pattern is not one of the 72 and cannot be: it is a depth-8 design (0 3 2 1 4 7 6 5 8), one of the **10 self-duals among the** $M(8) = 42$ seven-circuit mazes, whereas every depth-12 level sequence has thirteen entries and Cretan's has nine — the two censuses the L3 row below states correctly, and the sentence you are reading is the corrected form (2026-08-25). The Jericho design (0 3 4 5 2 1 6 7) is a valid maze that is *not* self-dual, which is what makes the property a discriminating one rather than a triviality of the construction. And because the space is enumerable, the null hypothesis is, for once, **exact rather than surrogate-approximated**: against 220 uniform draws from the full design space, built and measured identically, Chartres sits at the 22nd percentile on its scaling exponent, the 40th on its multifractal width, and the 52nd on its scattering statistic — squarely inside the null on every axis. Chartres is a *typical* valid labyrinth, distinguished by its symmetry class, not by any statistical excess. The methods moral generalizes well beyond mazes and is stated here because §6 will need its weaker cousin: where a design space can be enumerated, an approximate surrogate null is the wrong instrument — the exact null exists, and it is the one to use.

The same record also settles which mutual property the labyrinth carries, and the answer is a boundary this paper's funnel should be honest about. Read as a 1-D radius series and put through the estimators of §6, the labyrinth is the **anti-cascade**: its detrended-fluctuation exponent is scalebound ($\alpha \approx 1.76$ with a strong crossover at the circuit scale, $\alpha \approx 1.98 \rightarrow 1.49$), where a Gothic-ornament cascade transect sits in the long-range-correlated band without a crossover ($\alpha \approx 1.01$); its level-symbol string is fully described by twelve runs — one per circuit walked — so its LZ76 phrase count is 13 at *any* sampling resolution, where a matched-length cascade transect's phrase count grows with the number of samples taken ($c \approx 2,200$ at $n = 12,109$): the labyrinth's description length is set by its circuit count, not by its arc length (restated 2026-08-25 — the sealed record's *normalized* figures, 0.004 against an ornament transect's 0.71, are a sampling-density artifact and are not quoted here; see the L5 row); and its two-point multifractal width falls *below* its surrogate band. One estimator does fire — the scattering statistic reads a positive excess — and the enumerable null is precisely what exposes that reading as an artifact: the excess is generic to 100 % of valid labyrinths (the path's isolated turns, not any cascade), the reads-high-on-isolated-singularities failure mode that the program's estimator audits had already catalogued. So the old tag that architecture is frozen music survives only with its scope drawn: a design doctrine freezes into measurable cascade structure where it is *recursive and multiplicative* — ornament, the diffuser relief — and not where it is *combinatorial and topological*, as the labyrinth is: maximal path length at zero measure heterogeneity, the exact dual of a cascade. The labyrinth carries this section's mutual property, exact finite symmetry, in its purest pre-modern form, and carries none of §6's; the two properties are genuinely distinct, and a single stone floor can hold one and measurably lack the other. Its pre-registered ledger, sealed row by row (Atlas 2026e):

register row	pre-stated expectation	measured
L1	no width excess over matched IAAFT surrogates	confirmed (width $z = -6.3$, below the band); the scattering statistic's $+7.8$ excess is the isolated-singularity artifact, generic to 100 % of the enumerable null
L2	scaling exponent in the scalebound band, not ornament's LRC band	confirmed ($\alpha = 1.76$, crossover $\Delta = 0.49$; Cretan 1.67; ornament control $\alpha = 1.01$, no crossover)
L3	self-duality is exact (complement = reversal), the musical retrograde's $\mathbb{Z}_2$	confirmed by arithmetic; census 10/42 at depth 8, 72/1,828 at depth 12; Jericho not self-dual
L4	walker dwell integrates the local measure (the promenade hypothesis)	out of scope — no instrumented labyrinth-walking data exists anywhere
L5	level string more compressible than any matched ornament transect	**restated 2026-08-25 — the resolution-invariant part only.** The level string is twelve runs, LZ76 phrase count $c = 13$, at *every* sampling step, while a matched-length cascade transect's phrase count grows with the sample count ($c \approx 2,200$ at $n = 12,109$). The sealed *normalized* comparison (0.004 vs 0.71) is a sampling-density artifact: c is pinned at 13 by the twelve-run staircase while the reported value is $c/(n/\log_a n)$, so the same object traverses 0.001–0.71 as the sampling step varies, and a genuine MRW cascade coarse-grained to the labyrinth's own twelve run lengths scores bit-identically. The "beats all 40 draws" claim is withdrawn; the anti-cascade verdict rests on L2 and L6
L6	Chartres is not special against the exact enumerable null	confirmed (22nd / 40th / 52nd percentile on exponent / width / scattering; $M(12) = 1,828$)

Two honesty flags travel with the coda, in the spirit of §1's method note, both verified in the record. The ritual most often attached to labyrinths is documented at *Auxerre*, not Chartres: a 1396 chapter decree regulates the

Easter *pelota* ceremony danced on the Auxerre labyrinth (preserved through Lebeuf's eighteenth-century history of the diocese); a dance on the Chartres labyrinth is a modern assumption with no medieval documentation, and this paper does not repeat it. And the musical connection claimed here is the **cadence-level formalism only** — the exact $\mathbb{Z}_2$ shared by retrograde and self-duality — never the romantic pendulum of labyrinth-as-melody. The coda's claim is narrow, arithmetic, and checkable, which is, by this point in the paper, the only kind of claim the tradition has earned the right to make. Every number in this coda is read from that sealed record — Atlas 2026e, "The Labyrinth Arm (the Anti-Cascade): x7, Sealed Record v6 of the Feno Program," Zenodo DOI 10.5281/zenodo.22180454 (published 2026-08-31) — which carries the enumerable null, the L1–L6 register above, the regeneration commands and the seeds. The DOI was reserved against a private draft on 2026-08-30 and the record was **published 2026-08-31**: the DOI resolves, and the citation is live exactly as written.

6. Scaling: the mutual property the Fenophone occupies

The funnel now reaches its narrow end. The twentieth century's last contribution to the mutual property is **scale invariance** — self-similarity not between two magnitudes, nor within a single scale, but across a whole range of scales at once — and it arrives in music from two directions that the Fenophone is built to join.

The first direction is generative and, at first, loose. Mandelbrot's fractal geometry supplied the language of self-affinity, and his multiplicative cascades — coarse quantity redistributed to finer and finer scales by independent random weights, producing intermittency (Mandelbrot 1974) — supplied the mechanism. Voss and Clarke (1975, 1978) measured approximately 1/f power spectra in the audio and pitch fluctuations of music across genres, and then *inverted* the observation: they generated melodies from 1/f noise and reported that listeners preferred them to white-noise (uncorrelated) and brown-noise (random-walk) melodies. Gardner (1978) carried the idea to a wide audience, and a composition movement followed — Dodge's explicitly self-similar pieces, Bolognesi's self-similar dynamics, Pressing's nonlinear-map generators, Hsü and Hsü's fractal analyses, Díaz-Jerez's FractMus, the Zipf/power-law aesthetics of Manaris and colleagues (Dodge 1988; Bolognesi 1983; Pressing 1988; Hsü and Hsü 1990; Díaz-Jerez 2000; Manaris et al. 2005). But this tradition, for all its fertility, used fractal noise or a deterministic self-similar procedure as a *source of numbers to be mapped*, and its mathematical claim about the output — "fractal music," "1/f melody" — was rarely precise about the generator and almost never verified on what was emitted. It is the generative pole again, advancing an unchecked claim.

The second direction is descriptive and rigorous. The detrended fluctuation analysis of Peng and colleagues (1994) and its multifractal extension by Kantelhardt and colleagues (2002) — estimators built for DNA, heartbeat dynamics, and physiological time series — were turned on music and found not merely long-range correlation but genuine multifractality: a whole spectrum of scaling exponents rather than one, in note sequences (Su and Wu 2006), in the pitches of Bach's Inventions (Jafari, Pedram, and Hedayatifar 2007), and in Bach's Sinfonias treated as interacting voices (Telesca and Lovallo 2011) — and, more recently, in the timbre and playing-mode of stringed instruments (Banerjee et al. 2016; the multiscale fractal descriptors of Zlatintsi and Maragos 2013 separate instruments for recognition on the same principle) and in a soloist's performance-to-performance improvisation (Ghosh et al. 2018). Levitin, Chordia, and Menon (2012) showed that the rhythm spectra of a large notated corpus obey 1/f-family power laws with composer-dependent exponents; Hennig and colleagues (2011) found that even the involuntary microtiming of human performers is long-range correlated, and that listeners prefer such correlated fluctuation to white jitter (confirmed on a professional recording by Räsänen and colleagues 2015). The same multifractal "expressiveness" — a fluctuating balance of the predictable and the surprising across scales — has even been found beyond human music, in the rhythm of birdsong (Roeske, Kelty-Stephen, and Wallot 2018), which argues that it is a general mechanism of auditory engagement rather than an artifact of Western notation. The descriptive pole had established the target statistics with real estimators on real music — and, crucially, had also established the discipline's honesty test, since finite length and heavy tails alone can produce *apparent* multifractality with no nonlinear structure behind it (Bouchaud, Potters, and Meyer 2000), so that a measured width means nothing without a surrogate control.

What makes scale invariance the paper's terminal mutual property, rather than one more analogy, is that it is *genuinely shared*. The same multifractal, scale-free burst-and-lull that DFA and MFDFA find in music is the same structure the same estimators find in turbulence, in financial volatility, in heartbeat intervals, in gait. This is not a metaphor of the kind Xenakis drew when he called a string a gas molecule; it is one mathematical object recurring across physically unrelated systems, and it has a mature *generative* theory because one of those systems — finance — needed one. The lineage runs directly: Mandelbrot's turbulence cascade (1974), imported into finance as the Multifractal Model of Asset Returns (Mandelbrot, Fisher, and Calvet 1997), which was non-causal and grid-bound and therefore could not be run forward; and then Calvet and Fisher's repair, the Markov-switching multifractal (2001, 2004, 2008), which replaces the grid recursion with a Markov chain of volatility components switching at geometrically spaced rates, and is thereby *stationary, causal, Markov, estimable by maximum likelihood, and forecastable* (Lux 2008 gives a moment-based alternative). Real-time music generation needs exactly the same properties for exactly the same reason: an instrument must run indefinitely, respond at the next tick, and never privilege the moment it was switched on. The mutual property has narrowed all the way down — from the ancient ratio between two string-lengths to a single estimable stochastic process whose defining feature is a power-law symmetry across all time-scales — and it has narrowed onto an object that already exists, fully worked out, in a sister domain. That is the object the Fenophone is built on.

7. The Fenophone: the measured synthesis

The Fenophone can now be stated in one sentence that the preceding six sections have earned: it is the instrument at which the generative and descriptive poles fuse, on the mutual property of multifractal scaling, under engineering discipline. The four companion papers make the technical case; this section makes the historical one.

It takes the generative pole to its limit. The Fenophone's intensity process is, by construction, a Markov-switching multifractal — a product of k two-state multipliers switching at geometrically spaced rates, whose bands are routed into four musical lanes (dynamics, density, timbre, harmonic tension). Its primary controls are not *inspired by* the model; they *are* the model's own parameters — Intensity is the multiplier magnitude m_{hi}, Change Rate the slowest switch rate, Spread the time-scale span, Layers the count k — reached through monotone transfer functions, with two named and deliberate departures from the canonical econometric form (a log-symmetric multiplier, so that neutrality is geometric and "loudness never drifts with the Intensity setting"; and a span-parameterized rather than count-parameterized rate ladder, so that adding layers refines resolution rather than manufacturing fast motion) (Atlas 2026a; 2026c). This is the direct heir of every generative episode in the history above: of the Pythagorean identity of music with number, of Rameau's derivation of a grammar from a physical model, of the serial composer whose controls are a row, of Xenakis whose controls are a distribution's parameters. The instrument's signature gesture completes the lineage — the player can "pin" a latent multiplier, freezing one state of the Markov process and conducting the music from its regime structure, an intervention on the model's own hidden state that only a generative core with an explicit, enumerable latent state could offer (Atlas 2026a).

It takes the descriptive pole and hardens it into a gate. The Fenophone does not assert that its output is multifractal or 1/f; it *measures* the emitted note stream on every build, with the same DFA and MFDFA estimators the descriptive tradition of §6 developed, and enforces the result in continuous integration. The measured landscape is specific and honestly bounded: per-scaffold median onset-stream exponents $\alpha \approx 0.695\text{–}0.770$ and velocity exponents $\approx 0.676\text{–}0.702$ (squarely in the persistent 1/f family, away from both the uncorrelated 0.5 and the random-walk 1.5 fixed points), every seed beating its order-destroyed shuffle null; multifractal widths $0.054\text{–}0.126$, exceeding matched IAAFT surrogate nulls on every seed for three of four measured packs, with the fourth — the least intermittent by design — reported as consistent with linear long memory at this length and *not* claimed as multifractal (Atlas 2026a). The honesty test that Bouchaud, Potters, and Meyer (2000) forced on the descriptive tradition is thus built into the gate: the width floor is documented as a degeneracy guard, not a multifractality certificate, and the multifractality evidence is the per-pack surrogate excess. A future engine or content change that whitens the output, collapses it to a bar-periodic loop, or flattens its width fails the build before any listener hears it. That the guarantee bites is a contemporary finding: turned on today's state-of-the-art neural music generators, the same estimators find their output misses the human multifractal signature systematically (Zhang, Sun, and Xiong 2025) — surface fidelity without the scale-invariant structure, which those authors then propose to chase with fractal-aware training penalties. The Fenophone reaches the property from the other side, generating from a multifractal process and enforcing the result rather than learning and hoping for it. The estimator has been promoted from an instrument of analysis to a condition of release — the descriptive pole's method turned into the generative pole's guarantee.

The fusion is the point, and it is old. Because the design target (a multifractal generative core) and the verification (multifractal estimators on the output) are the *same mathematical object*, the Fenophone realizes, in automated and quantitative form, the exact stance Ptolemy proposed and antiquity abandoned: reason proposes the structure, perception — here, measurement — confirms or refutes it, and the instrument is where the check happens. "Measured, never assumed" is that ancient synthesis made mechanical. The precondition that makes it possible is one the history could not supply: determinism. The Fenophone's event layer is integer and fixed-point and bit-identical across every platform, so that a statistical claim over a pinned seed panel is a *deterministic function of the repository* — the reproducibility that Xenakis's ST program and Koenig's Project programs aspired to but could not guarantee (their outputs varied with the machine), and the reason a distributional gate can be strict without flaking (Atlas 2026b). Change ringing's decidable "every permutation exactly once" has become detrended fluctuation analysis's "median α above the published floor"; the integer-exact, exhaustively verified combinatorial structure of the bell-tower has become an integer-exact, statistically verified generative process. The monochord and the resonator — instruments that were also apparatus — have a descendant that not only sounds the mathematics but certifies it.

How it extends the tradition, precisely. Three ways. First, in what it imports. Where the fractal-music movement borrowed 1/f noise as a source, and Xenakis borrowed physics as metaphor, the Fenophone imports a *mature, estimable, invertible generative model from a sister domain that genuinely shares the mutual property*. The MSM is not an analogy to musical intensity; it is a working model of volatility, and volatility and musical intensity share the specific structure of scale-free burst-and-lull as a matter of measurement, not of poetry. Because the imported model is estimable and invertible, the extension runs both ways: the same estimation theory that fits the MSM to markets can, in principle, fit it to musical corpora and read composers off the parameter plane, closing the loop from generation back to description (Atlas 2026c; 2026d). This is the disciplined modern successor to *musica mundana* — the ancient claim that one law holds across the scales of the cosmos — with the metaphysics replaced by a measured, enforced invariant and the "cosmos" replaced by the honest observation that turbulence, markets, and music share a computable statistical law. Second, in what it closes: the two gaps this paper has tracked — the post-hoc formalization gap of the symmetry program and the assumed-not-measured gap of the stochastic program — are both shut by the same device, an estimator that is also a release gate. Third, in what it refuses to conflate. The Fenophone is explicit that the model supplies *motion with structure at every scale* while musical taste — consonance, voice-leading, groove, the register architecture — is authored human data in a versioned "scaffold," and that the two are separately guaranteed: the invariant suite proves a pack safe and structurally alive; a human listening panel judges whether it is good (Atlas 2026a; 2026b). The tradition repeatedly conflated these — Rameau's "chord of nature," the serialists' assumption that constructed structure must be heard structure, the fractalists' slide from "1/f generator" to "music listeners prefer." The Fenophone keeps the model's provable claims and the ear's unprovable ones apart on purpose.

What it does not settle. The paper's oldest dialectic survives intact, and the Fenophone is honest that it does. Everything above concerns the *number* side: that the output carries measured, surrogate-controlled multifractal scaling is established by construction and by continuous integration, independent of any listener. Whether that structure is *heard* — whether multifractality is perceptible beyond the power spectrum, whether the slow band is felt as tension, whether latent regime-flips are experienced as musical form — is the open Aristoxenian question, and the Fenophone answers it not with assertion but with a specified, falsifiable perceptual programme whose stimuli are the instrument's own bit-reproducible state vectors (Atlas 2026c). This is the honest modern form of Ptolemy's truce: the reason side is made rigorous and the ear side is made *testable*, rather than assumed. Aristoxenus and Vincenzo Galilei demanded that number-claims about music be validated against the sounding phenomenon; the Fenophone grants the demand, and leaves the ear the final vote.

8. Conclusion

The relationship between music and mathematics is often narrated as a series of enthusiasms — Pythagoras's ratios, Kepler's spheres, Schoenberg's rows, Xenakis's distributions, the fractalists' 1/f. Read as a single story, it is something more disciplined: a slow convergence of the mutual property, and a long failure to fuse the two ways of using it. The property narrows from **ratio** (two magnitudes) through **periodicity and the harmonic spectrum** (one scale) through **finite-group symmetry** (exact discrete invariance) and **the probability distribution** (the generating law) to **scale-invariant multifractality** (a power-law symmetry across all scales, shared with turbulence and markets and the body). And the two modes of use — generate from the model, measure from the sound — stay apart, so that the generative tradition keeps advancing mathematical claims it never checks on the output, and the descriptive tradition keeps measuring music it never builds.

The Fenophone is the point where the funnel's narrow end and the two modes meet. It generates from a mathematical model whose parameters are its controls, and it measures the mathematical property of its own output and enforces it — the same estimator as design target and as release gate, on the mutual property to which the whole history has been narrowing. In doing so it makes rigorous a synthesis first proposed by Ptolemy and abandoned for two thousand years: reason proposes the structure; measurement confirms or refutes it; the instrument is where the check happens. The monochord divided a string to make a ratio audible and checkable. The Fenophone runs a cascade to make a scaling law audible and checkable. Between those two instruments lies the history this paper has tried to tell — and the claim it defends is that they are the same kind of object, twenty-five centuries apart: a music-maker that is also an apparatus, holding a mathematical claim about music open to refutation, and, for the first time, doing so automatically, on every build, before anyone listens.

References

Primary historical works are cited inline by title and date (Plato's Timaeus *; Ptolemy's* Harmonics *; Kepler's* Harmonices Mundi *, 1619; Vincenzo Galilei's* Dialogo *, 1581; Mersenne's* Harmonie universelle *, 1636; Rameau's* Traité *, 1722; Euler's* Tentamen *, 1739; Helmholtz's* Sensations of Tone *, 1863; and the like), and the standard critical editions and English translations used for them are entered in the list below under the historical author's name. The remainder of the list gives the scholarly and secondary sources and companion papers cited author–date.*

Alembert, J. le Rond d'. 1747. "Recherches sur la courbe que forme une corde tendue mise en vibration." *Histoire de l'Académie Royale des Sciences et Belles-Lettres de Berlin* 3.

Aristoxenus. 1902. *The Harmonics of Aristoxenus (Elementa Harmonica).* Edited and translated by H. S. Macran. Oxford: Clarendon Press. (Also translated in Barker 1989.)

Atlas, E. T. 2026a. "The Fenophone: A Musical Instrument Built on the Markov-Switching Multifractal, with CI-Verified Spectral Invariants." Fenophone research series, paper 1. Zenodo DOI 10.5281/zenodo.22180423 (published 2026-08-31).

Atlas, E. T. 2026b. "Theorems as Instrument Design: Formal Guarantees and Statistical Contracts in a Generative Musical Instrument." Fenophone research series, paper 2. Zenodo DOI 10.5281/zenodo.22180427 (published 2026-08-31).

Atlas, E. T. 2026c. "The Markov-Switching Multifractal as a Generative Model of Musical Intensity: An Instrument, Measurements, and a Research Programme." Fenophone research series, paper 3. Zenodo DOI 10.5281/zenodo.22180429 (published 2026-08-31).

Atlas, E. T. 2026d. "From Latent Cascade to Note Stream: How Scaling Exponents Survive Musical Quantization." Fenophone research series, paper 4. Zenodo DOI 10.5281/zenodo.22180431 (published 2026-08-31).

Atlas, E. T. 2026e. "The Labyrinth Arm (the Anti-Cascade): x7, Sealed Record v6 of the Feno Program." Feno program record, 2026-08-22. Zenodo DOI 10.5281/zenodo.22180454 (published 2026-08-31). Regeneration commands and seeds in-record.

Babbitt, M. 1962. "Twelve-Tone Rhythmic Structure and the Electronic Medium." *Perspectives of New Music* 1 (1): 49–79.

Banerjee, A., S. Sanyal, T. Guhathakurata, R. Sengupta, and D. Ghosh. 2016. "Categorization of Stringed Instruments with Multifractal Detrended Fluctuation Analysis." arXiv:1601.07709.

Barker, A. 1989. *Greek Musical Writings, Vol. II: Harmonic and Acoustic Theory.* Cambridge: Cambridge University Press.

Barker, A. 2007. *The Science of Harmonics in Classical Greece.* Cambridge: Cambridge University Press.

Benson, D. J. 2007. *Music: A Mathematical Offering.* Cambridge: Cambridge University Press.

Bernoulli, D. 1753. “Réflexions et éclaircissemens sur les nouvelles vibrations des cordes exposées dans les mémoires de l’Académie de 1747 et 1748.” *Histoire de l’Académie Royale des Sciences et Belles-Lettres de Berlin* 9.

Bolognesi, T. 1983. “Automatic Composition: Experiments with Self-Similar Music.” *Computer Music Journal* 7 (1): 25–36.

Bouchaud, J.-P., M. Potters, and M. Meyer. 2000. “Apparent Multifractality in Financial Time Series.” *European Physical Journal B* 13 (3): 595–599.

Bower, C. M., trans. 1989. *Boethius, Fundamentals of Music (De institutione musica).* New Haven: Yale University Press.

Burkert, W. 1972. *Lore and Science in Ancient Pythagoreanism.* Translated by E. L. Minar. Cambridge, MA: Harvard University Press. (German original 1962.)

Calvet, L. E., and A. J. Fisher. 2001. “Forecasting Multifractal Volatility.” *Journal of Econometrics* 105 (1): 27–58.

Calvet, L. E., and A. J. Fisher. 2004. “How to Forecast Long-Run Volatility: Regime Switching and the Estimation of Multifractal Processes.” *Journal of Financial Econometrics* 2 (1): 49–83.

Calvet, L. E., and A. J. Fisher. 2008. *Multifractal Volatility: Theory, Forecasting, and Pricing.* Burlington, MA: Academic Press.

Christensen, T. 1987. “Eighteenth-Century Science and the *Corps Sonore*: The Scientific Background to Rameau’s Principle of Harmony.” *Journal of the American Musicological Society* 40 (1): 23–50.

Cohen, J. E. 1962. “Information Theory and Music.” *Behavioral Science* 7 (2): 137–163.

Crans, A. S., T. M. Fiore, and R. Satyendra. 2009. “Musical Actions of Dihedral Groups.” *The American Mathematical Monthly* 116 (6): 479–495.

Díaz-Jerez, G. 2000. “Algorithmic Music: Using Mathematical Models in Music Composition (the FractMus System).” D.M.A. thesis, Manhattan School of Music.

Dirichlet, P. G. L. 1829. “Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données.” *Journal für die reine und angewandte Mathematik* 4: 157–169.

Dodge, C. 1988. “Profile: A Musical Fractal.” *Computer Music Journal* 12 (3): 10–14.

Doornbusch, P. 2004. “Computer Sound Synthesis in 1951: The Music of CSIRAC.” *Computer Music Journal* 28 (1): 10–25.

Duckworth, R., and F. Stedman. 1668. *Tintinnalogia: or, the Art of Ringing.* London.

Duffin, R. W. 2006. “Just Intonation in Renaissance Theory and Practice.” *Music Theory Online* 12 (3).

Ellis, A. J. 1885. “On the Musical Scales of Various Nations.” *Journal of the Society of Arts* 33: 485–527.

Euler, L. 1739. *Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae.* St. Petersburg: Academy of Sciences. Reprinted in *Opera Omnia*, series III, volume 1. Leipzig: Teubner, 1926.

Fauvel, J., R. Flood, and R. Wilson, eds. 2003. *Music and Mathematics: From Pythagoras to Fractals.* Oxford: Oxford University Press.

Forte, A. 1973. *The Structure of Atonal Music.* New Haven: Yale University Press.

Fourier, J. 1822. *Théorie analytique de la chaleur.* Paris: Firmin Didot. Translated by A. Freeman as *The Analytical Theory of Heat.* Cambridge: Cambridge University Press, 1878. Reprint, New York: Dover, 1955.

Galilei, G. 1638. *Discorsi e dimostrazioni matematiche intorno a due nuove scienze.* Leiden: Elzevir. Translated by S. Drake as *Two New Sciences.* Madison: University of Wisconsin Press, 1974.

Galilei, V. 1581. *Dialogo della musica antica et della moderna.* Florence: Marescotti. Translated by C. V. Palisca as *Dialogue on Ancient and Modern Music.* New Haven: Yale University Press, 2003.

Gardner, M. 1978. “Mathematical Games: White and Brown Music, Fractal Curves and One-over-f Fluctuations.” *Scientific American* 238 (4): 16–32.

Ghosh, D., R. Sengupta, S. Sanyal, and A. Banerjee. 2018. “Improvisation—A New Approach of Characterization.” In *Musicality of Human Brain through Fractal Analytics*, chap. 9, 185–212. Singapore: Springer.

Helmholtz, H. von. 1863. *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik.* Braunschweig: Vieweg. Translated by A. J. Ellis as *On the Sensations of Tone as a Physiological Basis for the Theory of Music.* 2nd English edition, from the 4th German edition, London: Longmans, Green, 1885. Reprint, New York: Dover, 1954.

Hennig, H., R. Fleischmann, A. Fredebohm, Y. Hagmayer, J. Nagler, A. Witt, F. J. Theis, and T. Geisel. 2011. “The Nature and Perception of Fluctuations in Human Musical Rhythms.” *PLoS ONE* 6 (10): e26457.

Hiller, L. A., and L. M. Isaacson. 1959. *Experimental Music: Composition with an Electronic Computer.* New York: McGraw-Hill.

Hook, J. 2007. “Why Are There Twenty-Nine Tetrachords? A Tutorial on Combinatorics and Enumeration in Music Theory.” *Music Theory Online* 13 (4).

Hsü, K. J., and A. J. Hsü. 1990. “Fractal Geometry of Music.” *Proceedings of the National Academy of Sciences* 87 (3): 938–941.

Jafari, G. R., P. Pedram, and L. Hedayatifar. 2007. “Long-Range Correlation and Multifractality in Bach’s Inventions Pitches.” *Journal of Statistical Mechanics: Theory and Experiment* 2007 (04): P04012.

Kantelhardt, J. W., S. A. Zschiegner, E. Koscielny-Bunde, S. Havlin, A. Bunde, and H. E. Stanley. 2002. “Multifractal Detrended Fluctuation Analysis of Nonstationary Time Series.” *Physica A* 316 (1–4): 87–114.

Kepler, J. 1619. *Harmonices mundi libri V.* Linz: Plank. Translated by E. J. Aiton, A. M. Duncan, and J. V. Field as *The Harmony of the World.* Philadelphia: American Philosophical Society, 1997.

Levitin, D. J., P. Chordia, and V. Menon. 2012. “Musical Rhythm Spectra from Bach to Joplin Obey a 1/f Power Law.” *Proceedings of the National Academy of Sciences* 109 (10): 3716–3720.

Lewin, D. 1987. *Generalized Musical Intervals and Transformations.* New Haven: Yale University Press.

Lux, T. 2008. “The Markov-Switching Multifractal Model of Asset Returns: GMM Estimation and Linear Forecasting of Volatility.” *Journal of Business & Economic Statistics* 26 (2): 194–210.

Lynch, P. 2018. “Euler’s Degree of Agreeableness for Musical Chords.” *ThatsMaths* (essay). https://thatsmaths.com/2018/08/09/eulers-degree-of-agreeableness-for-musical-chords/.

Mandelbrot, B. B. 1974. “Intermittent Turbulence in Self-Similar Cascades: Divergence of High Moments and Dimension of the Carrier.” *Journal of Fluid Mechanics* 62 (2): 331–358.

Mandelbrot, B. B., A. J. Fisher, and L. E. Calvet. 1997. “A Multifractal Model of Asset Returns.” Cowles Foundation Discussion Paper No. 1164, Yale University.

Manaris, B., J. Romero, P. Machado, D. Krehbiel, T. Hirzel, W. Pharr, and R. B. Davis. 2005. “Zipf’s Law, Music Classification, and Aesthetics.” *Computer Music Journal* 29 (1): 55–69.

Mazzola, G. 2002. *The Topos of Music: Geometric Logic of Concepts, Theory, and Performance.* Basel: Birkhäuser.

Mersenne, M. 1636. *Harmonie universelle, contenant la théorie et la pratique de la musique.* Paris: Cramoisy. Facsimile reprint, Paris: Éditions du CNRS, 1963. The books on instruments translated by R. E. Chapman as *Harmonie universelle: The Books on Instruments.* The Hague: Nijhoff, 1957.

Messiaen, O. 1944. *Technique de mon langage musical.* Paris: Alphonse Leduc.

Meyer, L. B. 1957. “Meaning in Music and Information Theory.” *The Journal of Aesthetics and Art Criticism* 15 (4): 412–424.

Moles, A. 1966. *Information Theory and Esthetic Perception.* Translated by J. E. Cohen. Urbana: University of Illinois Press. (French original *Théorie de l’information et perception esthétique*, 1958.)

Ohm, G. S. 1843. “Ueber die Definition des Tones, nebst daran geknüpfter Theorie der Sirene und ähnlicher tonbildender Vorrichtungen.” *Annalen der Physik und Chemie* 59: 513–565.

Peng, C.-K., S. V. Buldyrev, S. Havlin, M. Simons, H. E. Stanley, and A. L. Goldberger. 1994. “Mosaic Organization of DNA Nucleotides.” *Physical Review E* 49 (2): 1685–1689.

Phillips, T. 2015. “Medieval Labyrinths and Their Sequences.” AMS Feature Column, October 2015. American Mathematical Society. (The SAT-maze conditions, level-sequence formalism, and design counts; companion enumeration pages at math.stonybrook.edu/~tony/mazes.)

Plato. 2000. *Timaeus.* Translated by D. J. Zeyl. Indianapolis: Hackett.

Plomp, R., and W. J. M. Levelt. 1965. “Tonal Consonance and Critical Bandwidth.” *Journal of the Acoustical Society of America* 38 (4): 548–560.

Pressing, J. 1988. “Nonlinear Maps as Generators of Musical Design.” *Computer Music Journal* 12 (2): 35–46.

Ptolemy. 2000. *Harmonics.* Translated with commentary by J. Solomon. Leiden: Brill. (Also translated in Barker 1989.)

Rameau, J.-P. 1722. *Traité de l’harmonie réduite à ses principes naturels.* Paris: Ballard. Translated by P. Gossett as *Treatise on Harmony.* New York: Dover, 1971.

Rameau, J.-P. 1726. *Nouveau système de musique théorique.* Paris: Ballard.

Rameau, J.-P. 1737. *Génération harmonique, ou Traité de musique théorique et pratique.* Paris: Prault.

Räsänen, E., O. Pulkkinen, T. Virtanen, M. Zollner, and H. Hennig. 2015. “Fluctuations of Hi-Hat Timing and Dynamics in a Virtuoso Drum Track of a Popular Music Recording.”

PLoS ONE 10 (6): e0127902.

Rayleigh, J. W. S. 1877. *The Theory of Sound.* 2 vols. London: Macmillan. Reprint of the 2nd edition, New York: Dover, 1945.

Roeske, T. C., D. Kelty-Stephen, and S. Wallot. 2018. "Multifractal Analysis Reveals Music-like Dynamic Structure in Songbird Rhythms." *Scientific Reports* 8: 4570.

Sauveur, J. 1701. "Système général des intervalles des sons, et son application à tous les systèmes et à tous les instruments de musique." *Mémoires de l'Académie Royale des Sciences* (Paris). Reprinted in *Collected Writings on Musical Acoustics (Paris 1700–1713)*, edited by R. Rasch. Utrecht: Diapason Press, 1984.

Schoenberg, A. 1975. *Style and Idea: Selected Writings of Arnold Schoenberg.* Edited by L. Stein, translated by L. Black. London: Faber and Faber. (Includes the 1941 lecture "Composition with Twelve Tones.")

Stedman, F. 1677. *Campanalogia: or, The Art of Ringing Improved.* London.

Su, Z.-Y., and T. Wu. 2006. "Multifractal Analyses of Music Sequences." *Physica D* 221 (2): 188–194.

Taylor, B. 1713. "De motu nervi tensi." *Philosophical Transactions of the Royal Society of London* 28.

Telesca, L., and M. Lovallo. 2011. "Revealing Competitive Behaviours in Music by Means of the Multifractal Detrended Fluctuation Analysis: Application to Bach's Sinfonias." *Proceedings of the Royal Society A* 467 (2134): 3022–3032.

Turner, R. S. 1977. "The Ohm-Seebeck Dispute, Hermann von Helmholtz, and the Origins of Physiological Acoustics." *British Journal for the History of Science* 10 (1): 1–24.

Voss, R. F., and J. Clarke. 1975. "'1/f Noise' in Music and Speech." *Nature* 258 (5533): 317–318.

Voss, R. F., and J. Clarke. 1978. "'1/f Noise' in Music: Music from 1/f Noise." *Journal of the Acoustical Society of America* 63 (1): 258–263.

Wheeler, G. F., and W. P. Crummett. 1987. "The Vibrating String Controversy." *American Journal of Physics* 55 (1): 33–37.

White, A. T. 1996. "Fabian Stedman: The First Group Theorist?" *The American Mathematical Monthly* 103 (9): 771–778.

Xenakis, I. 1971. *Formalized Music: Thought and Mathematics in Composition.* Bloomington: Indiana University Press. (Revised and expanded edition, Pendragon Press, Stuyvesant, NY, 1992.)

Zhang, K. K., Y. Sun, and H. Xiong. 2025. "Multifractal Comparison of Billboard and AI-Generated Music." In *Proceedings of the 33rd ACM International Conference on Multimedia (MM '25)*, 12419–12427. New York: ACM.

Zlatintsi, A., and P. Maragos. 2013. "Multiscale Fractal Analysis of Musical Instrument Signals with Application to Recognition." *IEEE Transactions on Audio, Speech, and Language Processing* 21 (4): 737–748.

Youngblood, J. E. 1958. "Style as Information." *Journal of Music Theory* 2 (1): 24–35.

Zarlino, G. 1558. *Le istitutioni harmoniche.* Venice. Facsimile reprint, New York: Broude Brothers, 1965. Part III translated by G. A. Marco and C. V. Palisca as *The Art of Counterpoint.* New Haven: Yale University Press, 1968.